DIMENSION OF ATTRACTORS AND INVARIANT SETS OF DAMPED WAVE EQUATIONS IN UNBOUNDED DOMAINS

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ABSTRACT. Under fairly general assumptions, we prove that every compact invariant set $\mathcal{I}$ of the semiflow generated by the semilinear damped wave equation

$$\begin{aligned} u_{tt} + \alpha u_t + \beta(x)u - \Delta u &= f(x, u), \quad (t, x) \in [0, +\infty[\times\Omega, \\ u &= 0, \qquad\qquad (t, x) \in [0, +\infty[\times\partial\Omega \end{aligned}$$

in $H^1_0(\Omega) \times L^2(\Omega)$ has finite Hausdorff and fractal dimension. Here Ω is a regular, possibly unbounded, domain in $\mathbb{R}^3$ and $f(x, u)$ is a nonlinearity of critical growth. The nonlinearity $f(x, u)$ needs not to satisfy any dissipativeness assumption and the invariant subset $\mathcal{I}$ needs not to be an attractor. If $f(x, u)$ is dissipative and $\mathcal{I}$ is the global attractor, we give an explicit bound on the Hausdorff and fractal dimension of $\mathcal{I}$ in terms of the structure parameters of the equation.

1. Introduction

In this paper we consider the damped wave equation

$$\begin{aligned} u_{tt} + \alpha u_t + \beta(x)u - \Delta u &= f(x, u), \quad (t, x) \in [0, +\infty[\times\Omega, \\ u &= 0, \qquad\qquad (t, x) \in [0, +\infty[\times\partial\Omega \end{aligned} \tag{1.1}$$

Here Ω is a regular (possibly unbounded) open set in $\mathbb{R}^3$, $\beta(x)$ is a potential such that the operator $-\Delta+\beta(x)$ is positive, and $f(x, u)$ is a nonlinearity of critical growth (i.e. of polynomial growth less than or equal to three). The assumptions on $\beta(x)$ and $f(x, u)$ will be made more precise in Section 2 below. Under such assumptions, equation (1.1) generates a local semiflow Π in the space $H^1_0(\Omega) \times L^2(\Omega)$. Suppose that the semiflow Π admits a compact invariant set $\mathcal{I}$ (i.e. $\Pi(t)\mathcal{I} = \mathcal{I}$ for all $t \geq 0$). We do not make any structure assumption on the nonlinearity $f(x, u)$ and therefore we do not assume that $\mathcal{I}$ is the global attractor of equation (1.1). Our aim is to prove that $\mathcal{I}$ has finite Hausdorff and fractal dimension and to give an explicit estimate of its dimension.

When Ω is a bounded domain and $f(x, u)$ satisfies suitable dissipativeness conditions, the existence of a finite dimensional compact global attractor for (1.1) is a classical achievement (see e.g. [11, 21] and the references therein).

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When Ω is unbounded, new difficulties arise due to the lack of compactness of the Sobolev embeddings. These difficulties can be overcome in several ways: by exploiting the *finite speed of propagation property* (e.g. in [6]), by introducing *weighted or uniform spaces* (see e.g. [22]), by developing suitable *tail-estimates* (see e.g. [15]).

Concerning the finite dimensionality of the attractor, in the *unbounded domain case* very few results are available. In [22] Zelik proved finite dimensionality of attractors in the context of uniform spaces, assuming that $\beta(x)$ is constant and $f(x,u)$ is independent of x and satisfies $f(u)u \leq 0$, $f'(u) \leq L$ for all $u \in \mathbb{R}$. The technique exploited by Zelik seems not to give explicit bounds for the dimension of the attractor. In [10], Karachalios and Stavrakakis considered an equation of the form

$$u_{tt} + \alpha u_t + \beta(x)u - g(x)^{-1}\Delta u = f(u) + h(x), \tag{1.2}$$

where $g(\cdot)$ is a positive function belonging to $L^\infty \cap L^{3/2}$. In this case the weight $g(x)^{-1}$ "forces" the operator $-g(x)^{-1}\Delta$ to have compact resolvent: the result then is achieved by exploiting directly the technique of *volume tracking* developed by Temam and other authors for bounded domains (see [21]).

In this paper we do not make any structure assumption on the nonlinearity $f(x,u)$. Our only assumption is that $\partial_u f(x,0)$ is non negative and belongs to $L^r(\Omega)$ for some $r > 3$. The positivity of $\partial_u f(x,0)$ is not a real restriction, because its negative part can be absorbed in $\beta(x)$. Under this assumption, we shall prove that $\mathcal{I}$ has finite Hausdorff and fractal dimension in the energy space $H^1_0(\Omega) \times L^2(\Omega)$. Also, we give an explicit estimate of the dimension of $\mathcal{I}$, in terms of the main parameters involved in the equation and of the quantity $\sup\{\|(u,v)\| \mid (u,v) \in \mathcal{I}\}$. In order to achieve our result, we shall exploit the technique of *volume tracking*, as expounded in [21]. However, we cannot apply directly the arguments of [21], since the operator $-\Delta + \beta(x)$ does not have compact resolvent. Indeed, in the *bounded domain case* (resp. in the *weighted Laplacian case* considered by Karachalios and Stavrakakis) the key point is that

$$\frac{1}{d}\sum_{j=1}^{d} \lambda_j^{-1} \to 0 \quad \text{as } d \to \infty, \tag{1.3}$$

where $(\lambda_j)_{j\in\mathbb{N}}$ is the sequence of the eigenvalues of $-\Delta$ (resp. of $-g(x)^{-1}\Delta$). In general the operator $-\Delta + \beta(x)$, when Ω is unbounded, does not satisfy such property, since it possesses a nontrivial essential spectrum and its eigenvalues below the bottom of the essential spectrum are finite or form a sequence which accumulate to the bottom the essential spectrum. Yet, a more accurate analysis shows that the numbers λ_j in (1.3) can be replaced by $\check{\lambda}_j$, where $(\check{\lambda}_j)_{j\in\mathbb{N}}$ is the sequence of the eigenvalues of the following *weighted eigenvalue problem*:

$$-\Delta\phi + \beta(x)\phi = \check{\lambda}\partial_u f(x,\bar{u}(x))^2\phi, \tag{1.4}$$

where $\bar{U} = (\bar{u},\bar{v}) \in \mathcal{I}$. It turns out that (1.4) has a pure point spectrum. Moreover, thanks to the Cwickel-Lieb-Rozenblum inequality, it is possible to determine the asymptotics of the sequence $(\check{\lambda}_j)_{j\in\mathbb{N}}$ independently of $\bar{U} \in \mathcal{I}$, and the result will follow.

The paper is organized as follows. In Section 2 we introduce notations, we state the main assumptions and we collect some preliminaries about the semiflow generated by equation (1.1). In Section 3 we recall the definition of Hausdorff and fractal dimension and we prove that any compact invariant set $\mathcal{I}$ of Π has finite Hausdorff and fractal dimension in $H^1_0(\Omega) \times L^2(\Omega)$. In Section 4 we specialize our result to the case of dissipative equations and we show that the dimensions of the attractors of (1.1) remain bounded as $\alpha \to \infty$.

2. Notation, preliminaries and remarks

Let $\sigma \geq 1$. We denote by $L^\sigma_{\mathrm{u}}(\mathbb{R}^N)$ the set of measurable functions $\omega\colon \mathbb{R}^N \to \mathbb{R}$ such that

$$|\omega|_{L^\sigma_{\mathrm{u}}} := \sup_{y \in \mathbb{R}^N} \left(\int_{B(y)} |\omega(x)|^\sigma \, dx \right)^{1/\sigma} < \infty,$$

where, for $y \in \mathbb{R}^N$, $B(y)$ is the open unit cube in $\mathbb{R}^N$ centered at y.

In this paper we assume throughout that $N = 3$, and we fix an open (possibly unbounded) set $\Omega \subset \mathbb{R}^3$.

Proposition 2.1. *Let $\sigma > 3/2$ and let $\omega \in L^\sigma_{\mathrm{u}}(\mathbb{R}^3)$. Set $\rho := 3/2\sigma$. Then, for every $\epsilon > 0$ and for every $u \in H^1_0(\Omega)$,*

$$\int_\Omega |\omega(x)||u(x)|^2 \, dx \leq |\omega|_{L^\sigma_{\mathrm{u}}} \left(\rho \epsilon M_B^2 |u|^2_{H^1} + (1-\rho)\epsilon^{-\rho/(1-\rho)} |u|^2_{L^2} \right), \tag{2.1}$$

where M_B the constant of the Sobolev embedding $H^1(B) \subset L^6(B)$ and B is the open unit cube in $\mathbb{R}^3$. Moreover, for every $u \in H^1_0(\Omega)$,

$$\int_\Omega |\omega(x)||u(x)|^2 \, dx \leq M_B^{2\rho} |\omega|_{L^\sigma_{\mathrm{u}}} |u|^{2\rho}_{H^1} |u|^{2(1-\rho)}_{L^2}. \tag{2.2}$$

Proof. See the proof of Lemma 3.3 in [16]. □

Let $\beta \in L^\sigma_{\mathrm{u}}(\mathbb{R}^3)$, with $\sigma > 3/2$. Let us consider the following bilinear form defined on the space $H^1_0(\Omega)$:

$$a(u, v) := \int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx + \int_\Omega \beta(x) u(x) v(x) \, dx, \quad u, v \in H^1_0(\Omega) \tag{2.3}$$

Our first assumption is the following:

Hypothesis 2.2. *There exists $\lambda_1 > 0$ such that*

$$\int_\Omega |\nabla u(x)|^2 \, dx + \int_\Omega \beta(x) |u(x)|^2 \, dx \geq \lambda_1 |u|^2_{L^2}, \quad u \in H^1_0(\Omega). \tag{2.4}$$

Remark 2.3. *Conditions on $\beta(x)$ under which Hypothesis 2.2 is satisfied are expounded e.g. in* [1, 2].

As a consequence of (2.4) and Proposition 2.1, we have:

Proposition 2.4. *There exist two positive constants* λ_0 *and* Λ_0 *such that*

$$\lambda_0|u|^2_{H^1} \leq \int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega \beta(x)|u(x)|^2\,dx \leq \Lambda_0|u|^2_{H^1}, \quad u \in H^1_0(\Omega). \tag{2.5}$$

The constants λ_0 *and* Λ_0 *can be computed explicitly in terms of* λ_1, M_B *and* $|\beta|_{L^\sigma_\mathrm{u}}$.

Proof. Cf Lemma 4.2 in [15] □

It follows from Proposition 2.4 that the bilinear form $a(\cdot,\cdot)$ defines a scalar product in $H^1_0(\Omega)$, equivalent to the standard one.

Notation 1. *From now on, we set* $\langle\cdot,\cdot\rangle_{H^1_0} := a(\cdot,\cdot)$ *and we denote by* $\|\cdot\|_{H^1_0}$ *the norm associated with* $\langle\cdot,\cdot\rangle_{H^1_0}$. *Also, we shall use the notation* $\|\cdot\|_{L^p}$ *to denote the* L^p*-norm in* $L^p(\Omega)$, $1 \leq p \leq \infty$.

Let $\mathbf{A}$ be the self-adjoint operator on $L^2(\Omega)$ defined by the differential operator $u \mapsto \beta(x)u - \Delta u$.

Then $\mathbf{A}$ generates a family X^κ, $\kappa \in \mathbb{R}$, of fractional power spaces with $X^{-\kappa}$ being the dual of X^κ for $\kappa \in]0,+\infty[$. For $\kappa \in]0,+\infty[$, the space X^κ is a Hilbert space with respect to the scalar product

$$\langle u,v\rangle_{X^\kappa} := \langle \mathbf{A}^\kappa u, \mathbf{A}^\kappa v\rangle_{L^2}, \quad u,v \in X^\kappa.$$

Also, the space $X^{-\kappa}$ is a Hilbert space with respect to the scalar product $\langle\cdot,\cdot\rangle_{X^{-\kappa}}$ dual to the scalar product $\langle\cdot,\cdot\rangle_{X^\kappa}$, i.e.

$$\langle u',v'\rangle_{X^{-\kappa}} = \langle R_\kappa^{-1}u', R_\kappa^{-1}v'\rangle_{X^\kappa}, \quad u,v \in X^{-\kappa},$$

where $R_\kappa \colon X^\kappa \to X^{-\kappa}$ is the Riesz isomorphism $u \mapsto \langle\cdot,u\rangle_{X^\kappa}$.

We make the following assumption:

Hypothesis 2.5. *The open set* Ω *is a uniformly* C^2 *domain in the sense of Browder* [3, p. 36].

As a consequence, by elliptic regularity we have that $D(-\Delta) = H^2(\Omega) \cap H^1_0(\Omega) \subset L^\infty(\Omega)$. In this situation, the assignment $u \mapsto \beta(x)u$ defines a relatively bounded perturbation of $-\Delta$ and therefore $D(-\Delta + \beta(x)) = H^2(\Omega) \cap H^1_0(\Omega)$. It follows that $X^\kappa \subset L^\infty(\Omega)$ for $\kappa > 3/4$ (see [8, Th. 1.6.1]).

We write

$$H_\kappa = X^{\kappa/2}, \quad \kappa \in \mathbb{R}.$$

Note that $H_0 = L^2(\Omega)$, $H_1 = H^1_0(\Omega)$, $H_{-1} = H^{-1}(\Omega)$ and $H_2 = H^2(\Omega) \cap H^1_0(\Omega)$.

For $\kappa \in \mathbb{R}$ the operator $\mathbf{A}$ induces a self-adjoint operator $\mathbf{A}_\kappa \colon H_{\kappa+2} \to H_\kappa$. In particular $\mathbf{A} = \mathbf{A}_0$. Moreover,

$$\langle u,v\rangle_{H^1_0} = \langle \mathbf{A}_0 u, v\rangle_{L^2}, \quad u \in D(\mathbf{A}_0),\ v \in H^1_0(\Omega).$$

For $\kappa \in \mathbb{R}$ set $Z_\kappa := H_{\kappa+1} \times H_\kappa$. For $\alpha > 0$ define the linear operator $\mathbf{B}_\kappa \colon Z_{\kappa+1} \to Z_\kappa$ by

$$\mathbf{B}_\kappa(u,v) := (v, -(\alpha v + \mathbf{A}_\kappa u)), \quad (u,v) \in Z_{\kappa+1}.$$

It follows that $\mathbf{B}_\kappa$ is m-dissipative on Z_κ (cf the proof of Prop. 3.6 in [16]). Therefore, by the Hille-Yosida-Phillips theorem (see e.g. [4]), $\mathbf{B}_\kappa$ is the infinitesimal generator of a C^0-semigroup $\mathbf{T}_\kappa(t)$, $t \in [0,+\infty[$, on Z_κ.

Given a function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$, we denote by $\hat{g}$ the Nemitski operator which associates with every function $u\colon \Omega \to \mathbb{R}$ the function $\hat{g}(u)\colon \Omega \to \mathbb{R}$ defined by

$$\hat{g}(u)(x) = g(x, u(x)), \quad x \in \Omega.$$

If $I \subset \mathbb{R}$, X is a normed spaces and if $u\colon I \to X$ is a function which is differentiable as a function into X then we denote its X-valued derivative by $(\partial_t \mid X)\, u$. Similarly, if X is a Banach space and $u\colon I \to X$ is integrable as a function into X, then we denote its X-valued integral by $\int_I u(t)\,(dt \mid X)$. If X and Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y. If $X = Y$ we write just $\mathcal{L}(X)$.

Hypothesis 2.6.

(1) *$f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is such that, for every $u \in \mathbb{R}$, $f(\cdot, u)$ is measurable and $f(\cdot, 0) \in L^2(\Omega)$;*

(2) *for a.e. $x \in \Omega$, $f(x, \cdot)$ is of class C^2, $\partial_u f(\cdot, 0) \in L^\infty(\Omega)$ and there exists a constants $C \geq 0$ such that*

$$|\partial_{uu} f(x, u))| \leq C(1 + |u|), \quad (x, u) \in \Omega \times \mathbb{R}.$$

The main properties of the Nemitski operator associated with f are collected in the following Proposition, whose proof is left to the reader.

Proposition 2.7. *Assume Hypothesis 2.6. Then $\hat{f}\colon H_0^1(\Omega) \to L^2(\Omega)$ is continuously differentiable, $D\hat{f}(u)[v](x) = \partial_u f(x, u(x))v(x)$ for u, $v \in H_0^1(\Omega)$, and there exists a positive constant $\tilde{C} > 0$ such that the following estimates hold:*

$$\|\hat{f}(u)\|_{L^2} \leq \tilde{C}(1 + \|u\|_{H_0^1}^3), \quad u \in H_0^1(\Omega) \tag{2.6}$$

$$\|D\hat{f}(u)\|_{\mathcal{L}(H_0^1, L^2)} \leq \tilde{C}(1 + \|u\|_{H_0^1}^2), \quad u \in H_0^1(\Omega) \tag{2.7}$$

$$\|D\hat{f}(u_1) - D\hat{f}(u_2)\|_{\mathcal{L}(H_0^1, L^2)} \leq \tilde{C}(1 + \|u_1\|_{H_0^1} + \|u_2\|_{H_0^1})\|u_1 - u_2\|_{H_0^1}, \qquad u_1, u_2 \in H_0^1(\Omega). \tag{2.8}$$

If $u \in H_0^1(\Omega)$ and $v \in L^2(\Omega)$, then $\widehat{\partial_u f}(u) \cdot v \in H^{-1}(\Omega)$ and the following estimates hold:

$$\|\widehat{\partial_u f}(u)\|_{\mathcal{L}(L^2, H^{-1})} \leq \tilde{C}(1 + \|u\|_{H_0^1}^2), \quad u \in H_0^1(\Omega) \tag{2.9}$$

$$\|\widehat{\partial_u f}(u_1) - \widehat{\partial_u f}(u_2)\|_{\mathcal{L}(L^2, H^{-1})} \leq \tilde{C}(1 + \|u_1\|_{H_0^1} + \|u_2\|_{H_0^1})\|u_1 - u_2\|_{H_0^1}, \qquad u_1, u_2 \in H_0^1(\Omega). \tag{2.10}$$

□

We consider the following semi-linear damped wave equation:

$$\begin{aligned} u_{tt} + \alpha u_t + \beta(x)u - \Delta u &= f(x, u), & (t, x) &\in [0, +\infty[\times\Omega, \\ u &= 0, & (t, x) &\in [0, +\infty[\times\partial\Omega \end{aligned} \tag{2.11}$$

with Cauchy data $u(0) = u_0$, $u_t(0) = v_0$.

We recall the following classical result (see e.g. Theorem II.1.3 in [7]):

Theorem 2.8. *Let X be a Banach space and let $B\colon D(B) \subset X \to X$ be the infinitesimal generator of a C^0-semigroup of linear operators $T(t)$, $t \in \mathbb{R}_+$. Consider the abstract Cauchy problem*

$$\begin{cases} \dot{u} = Bu(t) + f(t), \quad t \in \mathbb{R}_+ \\ u(0) = u_0 \end{cases} \tag{2.12}$$

Assume that $u_0 \in D(B)$ and that either

(1) *$f \in C(\mathbb{R}_+, X)$ takes values in $D(B)$ and $Bf \in C(\mathbb{R}_+, X)$, or*
(2) *$f \in C^1(\mathbb{R}_+, X)$.*

Then (2.12) has a unique solution $u \in C^1(\mathbb{R}_+)$ with values in $D(B)$. The solution is given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\, ds. \tag{2.13}$$

□

Using Theorem 2.8, we rewrite equation (2.11) as an integral evolution equation in the space $Z_0 = H_0^1(\Omega) \times L^2(\Omega)$, namely

$$(u(t), v(t)) = \mathbf{T}_0(t)(u_0, v_0) + \int_0^t \mathbf{T}_0(t-p)(0, \hat{f}(u(p)))\,(dp \mid Z_0). \tag{2.14}$$

Equation (2.14) is called the *mild formulation* of (2.11) and solutions of (2.14) are called *mild solutions* of (2.11). Note that by Proposition 2.1 the nonlinear operator $(u, v) \mapsto (0, \hat{f}(u))$ is Lipschitz continuous from Z_0 into itself. A classical Picard iteration argument shows that, if $(u_0, v_0) \in Z_0$, then (2.14) possesses a unique continuous maximal solution $(u(\cdot), v(\cdot))\colon [0, t_{\max}[\to Z_0$ (see Theor. 4.3.4 and Prop. 4.3.7 in [4]). We thus obtain a local semiflow on Z_0, which we denote by $\Pi(t)U_0$, $U_0 = (u_0, v_0) \in Z_0$, $t \in [0, t_{\max}(U_0)[$. Notice that the solution $(u(\cdot), v(\cdot))$ of (2.14) also satisfies

$$(u(t), v(t)) = \mathbf{T}_{-1}(t)(u_0, v_0) + \int_0^t \mathbf{T}_{-1}(t-p)(0, \hat{f}(u(p)))\,(dp \mid Z_{-1}). \tag{2.15}$$

Therefore, it follows from Theorem 2.8 that $(u(\cdot), v(\cdot))$ is continuously differentiable into Z_{-1} and

$$(\partial_t \mid Z_{-1})(u(t), v(t)) = \mathbf{B}_{-1}(u(t), v(t)) + (0, \hat{f}(u(t))). \tag{2.16}$$

In particular, one has

$$\begin{cases} (\partial_t \mid H_0)u(t) = v(t) \\ (\partial_t \mid H_{-1})v(t) = -\alpha v(t) - \mathbf{A}_{-1}u(t) + \hat{f}(u(t)) \end{cases} \tag{2.17}$$

Definition 2.9. *A function $(u(\cdot), v(\cdot))\colon \mathbb{R} \to Z_0$ is called a* full solution *of the semiflow Π generated by (2.14) iff, for every s, $t \in \mathbb{R}$, with $s \le t$, one has*

$$(u(t), v(t)) = \Pi(t-s)(u(s), v(s))$$

Definition 2.10. *A subset $\mathcal{I}$ of Z_0 is called* invariant *for the semiflow generated by (2.14) if for every $(u_0, v_0) \in \mathcal{I}$ there exists a full solution $(u(\cdot), v(\cdot))$ of (2.14) with $(u(0), v(0)) = (u_0, v_0)$ and $(u(t), v(t)) \in \mathcal{I}$ for all $t \in \mathbb{R}$.*

From now on we assume that $\mathcal{I} \subset Z_0$ is a compact invariant subset of the semiflow Π.

Notation 2. *If $\mathcal{B}$ is a Banach space such that $\mathcal{I} \subset \mathcal{B}$, we define*

$$|\mathcal{I}|_{\mathcal{B}} := \max\{\|u\|_{\mathcal{B}} \mid u \in \mathcal{I}\}. \tag{2.18}$$

We recall the following result:

Theorem 2.11 (cf Corollaries 2.10 and 2.13 in [13]). *Assume that Hypotheses 2.2, 2.5 and 2.6 are satisfied. Let $\mathcal{I} \subset Z_0$ be a compact invariant set of the semiflow generated by (2.14). Then $\mathcal{I}$ is a bounded subset of Z_1. Moreover, $|\mathcal{I}|_{Z_1}$ can be explicitly estimated in terms of $|\mathcal{I}|_{Z_0}$ and of the constants in Hypotheses 2.2 and 2.5.*

Let $\bar{U}_0 = (\bar{u}_0, \bar{v}_0) \in \mathcal{I}$, and let $\bar{U}(t) = (\bar{u}(t), \bar{v}(t))$, $t \in \mathbb{R}$, be the full bounded solution through $\bar{U}_0$. Given $H_0 = (h_0, k_0) \in Z_0$, let us denote by $\mathcal{U}(\bar{U}_0; t)H_0$ the mild solution of

$$(h(t), k(t)) = \mathbf{T}_0(t)(h_0, k_0) + \int_0^t \mathbf{T}_0(t-p)(0, \widehat{\partial_u f}(\bar{u}(p))h(p))\,(dp \mid Z_0). \tag{2.19}$$

Notice that $\mathcal{U}(\bar{U}_0; t)$ coincides with the restriction to Z_0 of the *evolution family* $\mathbf{U}_{-1}(t, s)$ generated in Z_{-1} by the family $\mathbf{B}_{-1} + \mathbf{C}_{-1}(t)$, $t \in \mathbb{R}$, where $\mathbf{C}_{-1}(t)(h, k) := (0, \widehat{\partial_u f}(\bar{u}(t))h)$ (see [13] and [9]).

A standard computation using Gronwall's inequality and Proposition 2.7 leads to the following:

Proposition 2.12. *For every $t \geq 0$,*

$$\sup_{\bar{U}_0 \in \mathcal{I}} \|\mathcal{U}(\bar{U}_0; t)\|_{\mathcal{L}(Z_0, Z_0)} < +\infty, \tag{2.20}$$

and

$$\lim_{\epsilon \to 0} \sup_{\substack{\bar{U}_1, \bar{U}_2 \in \mathcal{I} \\ 0 < \|\bar{U}_1 - \bar{U}_2\|_{Z_0} < \epsilon}} \frac{\|\Pi(t)(\bar{U}_2) - \Pi(t)(\bar{U}_1) - \mathcal{U}(\bar{U}_1; t)(\bar{U}_2 - \bar{U}_1)\|_{Z_0}}{\|\bar{U}_2 - \bar{U}_1\|_{Z_0}} = 0, \tag{2.21}$$

where $\bar{U}i = (\bar{u}_i, \bar{v}_i)$, $i = 1, 2, 3$. □

3. Dimension of invariant sets

Let $\mathcal{X}$ be a complete metric space and let $\mathcal{K} \subset \mathcal{X}$ be a compact set. For $d \in \mathbb{R}^+$ and $\epsilon > 0$ one defines

$$\mu_H(\mathcal{K}, d, \epsilon) := \inf \left\{ \sum_{i \in I} r_i^d \mid \mathcal{K} \subset \bigcup_{i \in I} B(x_i, r_i),\ r_i \leq \epsilon \right\}, \tag{3.1}$$

where the infimum is taken over all the finite coverings of $\mathcal{K}$ with balls of radius $r_i \leq \epsilon$. Observe that $\mu_H(\mathcal{K}, d, \epsilon)$ is a non increasing function of ϵ and d. The d-dimensional Hausdorff measure of $\mathcal{K}$ is by definition

$$\mu_H(\mathcal{K}, d) := \lim_{\epsilon\to 0} \mu_H(\mathcal{K}, d, \epsilon) = \sup_{\epsilon>0} \mu_H(\mathcal{K}, d, \epsilon). \tag{3.2}$$

One has:

(1) $\mu_H(\mathcal{K}, d) \in [0, +\infty]$;
(2) if $\mu_H(\mathcal{K}, \bar{d}) < \infty$, then $\mu_H(\mathcal{K}, d) = 0$ for all $d > \bar{d}$;
(3) if $\mu_H(\mathcal{K}, \bar{d}) > 0$, then $\mu_H(\mathcal{K}, d) = +\infty$ for all $d < \bar{d}$.

The Hausdorff dimension of $\mathcal{K}$ is the smallest d for which $\mu_H(\mathcal{K}, d)$ is finite, i.e.

$$\dim_H(\mathcal{K}) := \inf\{d > 0 \mid \mu_H(\mathcal{K}, d) = 0\}. \tag{3.3}$$

Now let $n_{\mathcal{K}}(\epsilon)$, $\epsilon > 0$, denote the minimum number of balls of $\mathcal{X}$ of radius ϵ which is necessary to cover $\mathcal{K}$. The fractal dimension of $\mathcal{K}$ is the number

$$\dim_F(\mathcal{K}) := \limsup_{\epsilon\to 0} \frac{\log n_{\mathcal{K}}(\epsilon)}{\log 1/\epsilon}. \tag{3.4}$$

There is a well developed technique to estimate the Hausdorff dimension of an invariant set of a map or a semigroup. We refer the reader e.g. to [21] and [11]. The geometric idea consists in tracking the evolution of a d-dimensional volume under the action of the linearization of the semigroup along solutions lying in the invariant set. One looks then for the smallest d for which any d-dimensional volume contracts asymptotically as $t \to \infty$.

We fix $\delta \in \mathbb{R}$ and we introduce a *change of coordinates* in the space Z_κ, $\kappa \in \mathbb{R}$, by

$$R_\delta \colon Z_\kappa \to Z_\kappa, \quad (u, v) \mapsto (u, v + \delta u).$$

The constant δ is to be fixed later. Clearly the transformation R_δ is linear, bounded and invertible, with inverse $R_\delta^{-1} = R_{-\delta}$. We define the semiflow

$$\Pi_\delta(t) := R_\delta \circ \Pi(t) \circ R_{-\delta}$$

and we set $\mathcal{I}_\delta := R_\delta \mathcal{I}$. Then $\mathcal{I}_\delta$ is a compact invariant set of Π_δ, it is bounded in Z_1, and $\dim \mathcal{I}_\delta = \dim \mathcal{I}$. For $\tilde{U}_0 \in \mathcal{I}_\delta$ and $t \geq 0$ we set

$$\mathcal{U}_\delta(\tilde{U}_0; t) := R_\delta \circ \mathcal{U}(R_{-\delta}\tilde{U}_0; t) \circ R_{-\delta}.$$

Then the conclusions of Proposition 2.12 hold with $\Pi(t)$, $\mathcal{I}$ and $\mathcal{U}(\bar{U}; t)$ replaced by $\Pi_\delta(t)$, $\mathcal{I}_\delta$ and $\mathcal{U}_\delta(\tilde{U}; t)$.

Let $\tilde{U}_0 = (\tilde{u}_0, \tilde{v}_0) \in \mathcal{I}_\delta$ and let $\tilde{U}(t) = (\tilde{u}(t), \tilde{v}(t)) = \Pi_\delta(t)\tilde{U}_0$. Let $\Phi_{0,i}$, $i = 1, \ldots, d$, be linearly independent elements of Z_0, $\Phi_{0,i} = (\phi_{0,i}, \psi_{0,i})$. Set $\Phi_i(t) := \mathcal{U}_\delta(\tilde{U}_0; t)\Phi_{0,i}$. We denote by $G(t)$ the square of the d-dimensional volume delimited by $\Phi_1(t)$, …, $\Phi_\delta(t)$, that is

$$G(t) := \|\Phi_1(t) \wedge \cdots \wedge \Phi_d(t)\|^2_{\wedge^d Z_0} = \det(\langle \Phi_i(t), \Phi_j(t)\rangle_{Z_0})_{ij}. \tag{3.5}$$

We need to find a differential equation satisfied by $G(t)$.

Lemma 3.1. *Let i and $j \in \{1, \dots, d\}$ be fixed. Then the function $t \mapsto \langle \Phi_i(t), \Phi_j(t)\rangle_{Z_0}$ is continuously differentiable, and*

$$\frac{d}{dt}\langle \Phi_i, \Phi_j\rangle_{Z_0} = -2\delta\langle \phi_i, \phi_j\rangle_{H_0^1} - 2(\alpha-\delta)\langle \psi_i, \psi_j\rangle_{L^2} \tag{3.6}$$
$$+ \delta(\alpha-\delta)(\langle \phi_i, \psi_j\rangle_{L^2} + \langle \psi_i, \phi_j\rangle_{L^2}) + (\langle \widehat{\partial_u f}(\tilde u(t))\phi_i, \psi_j\rangle_{L^2} + \langle \psi_i, \widehat{\partial_u f}(\tilde u(t))\phi_j\rangle_{L^2}).$$

Proof. First set $\bar U_0 := R_{-\delta}\tilde U_0$, $\bar U(t) := R_{-\delta}\tilde U(t)$, $\Theta_{0,l} = (\theta_{0,l}, \chi_{0,l}) := R_{-\delta}\Phi_{0,l}$, $l = i, j$, and $\Theta_l(t) = (\theta_l(t), \chi_l(t)) := R_{-\delta}\Phi_l(t)$, $l = i, j$. Notice that $\Theta_l(t) = \mathcal{U}(\bar U_0; t)\Theta_{0,l}$, $l = i, j$. It follows that $\langle \Phi_i(t), \Phi_j(t)\rangle_{Z_0} = \langle R_\delta \Theta_i(t), R_\delta\Theta_j(t)\rangle_{Z_0}$. Now we shall apply Theorem 2.6 in [16]. Set:

- $Z := Z_0 \oplus Z_0$;
- $T(t) := \mathbf{T}_0(t) \oplus \mathbf{T}_0(t)$;
- $B := \mathbf{B}_0 \oplus \mathbf{B}_0$;
- $g(s) = (0, \widehat{\partial_u f}(\tilde u(t))\theta_i(t)) \oplus (0, \widehat{\partial_u f}(\tilde u(t))\theta_j(t))$;
- $z(t) = \Theta_i(t) \oplus \Theta_j(t)$;
- $V(U_1, U_2) := \langle R_\delta U_1, R_\delta U_2\rangle_{Z_0}$

A standard computation shows that V is Fréchet differentiable in Z; moreover, for $U_i \oplus U_j \in D(B)$ and $H_i \oplus H_j \in Z$,

$$\begin{aligned}
DV(U_i \oplus U_j)[B(U_i\oplus U_j) + H_i \oplus H_j] &= \langle v_i + h_i, u_j\rangle_{H_0^1} + \delta\langle v_i + h_i, \delta u_j + v_j\rangle_{L^2} \\
&\quad + \langle -\alpha v_i + k_i, \delta u_j + v_j\rangle_{L^2} - \langle \mathbf{A}_0 u_i, \delta u_j + v_j\rangle_{L^2} + \langle u_i, v_j + h_j\rangle_{H_0^1} \\
&\quad + \delta\langle \delta u_i + v_i, v_j + h_j\rangle_{L^2} + \langle \delta u_i + v_i, -\alpha v_j + k_j\rangle_{L^2} - \langle \delta u_i + v_i, \mathbf{A}_0 u_j\rangle_{L^2} \\
&= -2\delta\langle u_i, u_j\rangle_{H_0^1} + (\langle h_i, u_j\rangle_{H_0^1} + \langle u_i, h_j\rangle_{H_0^1}) + (\langle k_i, \delta u_j + v_j\rangle_{L^2} + \langle \delta u_i + v_i, k_j\rangle_{L^2}) \\
&\quad + (\langle \delta(v_i + h_i) - \alpha v_i, \delta u_j + v_j\rangle_{L^2} + \langle \delta u_i + v_i, \delta(v_j + h_j) - \alpha v_j\rangle_{L^2})
\end{aligned}$$

where $U_l = (u_l, v_l)$ and $H_l = (h_l, k_l)$, $l = i, j$. It follows from Theorem 2.6 in [16] that

$$\begin{aligned}
\frac{d}{dt}\langle \Phi_i, \Phi_j\rangle_{Z_0} = \frac{d}{dt}V(\Theta_i, \Theta_j) &= -2\delta\langle \theta_i, \theta_j\rangle_{H_0^1} + (\langle(\delta-\alpha)\chi_i, \delta\theta_j + \chi_j\rangle_{L^2} \\
&\quad + \langle \delta\theta_i + \chi_i, (\delta-\alpha)\chi_j\rangle_{L^2} + (\langle \widehat{\partial_u f}(\tilde u(t))\theta_i, \delta\theta_j + \chi_j\rangle_{L^2} + \langle \delta\theta_i + \chi_i, \widehat{\partial_u f}(\tilde u(t))\theta_j\rangle_{L^2} \\
&= -2\delta\langle \phi_i, \phi_j\rangle_{H_0^1} - 2(\alpha-\delta)\langle \psi_i, \psi_j\rangle_{L^2} + \delta(\alpha-\delta)(\langle \phi_i, \psi_j\rangle_{L^2} + \langle \psi_i, \phi_j\rangle_{L^2}) \\
&\quad + (\langle \widehat{\partial_u f}(\tilde u(t))\phi_i, \psi_j\rangle_{L^2} + \langle \psi_i, \widehat{\partial_u f}(\tilde u(t))\phi_j\rangle_{L^2})
\end{aligned}$$

and the proof is completed. □

Let $\tilde U = (\tilde u, \tilde v) \in \mathcal{I}_\delta$ and let Σ_d be a d-dimensional subspace of Z_0. On Σ_d we define a self-adjoint operator $\mathbf{B}_{\tilde U, \Sigma_d, \delta}$ by

$$\langle \mathbf{B}_{\tilde U, \Sigma_d, \delta}(u, v), (\xi, \eta)\rangle_{Z_0} := -2\delta\langle u, \xi\rangle_{H_0^1} - 2(\alpha-\delta)\langle v, \eta\rangle_{L^2} \tag{3.7}$$
$$+ \delta(\alpha-\delta)(\langle u, \eta\rangle_{L^2} + \langle v, \xi\rangle_{L^2}) + (\langle \widehat{\partial_u f}(\tilde u)u, \eta\rangle_{L^2} + \langle v, \widehat{\partial_u f}(\tilde u)\xi\rangle_{L^2}),$$

for (u, v) and $(\xi, \eta) \in \Sigma_d$.

Now let $\tilde{U}_0$, $\tilde{U}(t)$, $\Phi_{0,i}$ and $\Phi_i(t)$, $i = 1, \dots, d$, and $G(t)$ be as above. We set $\Sigma_d(t) := \operatorname{span}(\Phi_1(t), \dots, \Phi_d(t))$ and we define a $(d \times d)$- matrix $(b_{il}(t))_{il}$ such that

$$\mathbf{B}_{\tilde{U}(t),\Sigma_d(t),\delta}\Phi_i(t) = \sum_{l=1}^{d} b_{il}(t)\Phi_l(t).$$

It follows from Lemma 3.1 that

$$\frac{d}{dt}\langle \Phi_i(t), \Phi_j(t)\rangle_{Z_0} = \langle \mathbf{B}_{\tilde{U}(t),\Sigma_d(t),\delta}\Phi_i(t), \Phi_j(t)\rangle_{Z_0} = \sum_{l=1}^{d} b_{il}(t)\langle \Phi_l(t), \Phi_j(t)\rangle_{Z_0}. \tag{3.8}$$

A straightforward computation now shows that

$$\frac{d}{dt}G(t) = \left(\sum_{i=1}^{d} b_{ii}(t)\right) G(t) = \operatorname{Tr}(\mathbf{B}_{\tilde{U}(t),\Sigma_d(t),\delta})G(t). \tag{3.9}$$

Therefore we get:

$$\|\Phi_1(t) \wedge \dots \wedge \Phi_d(t)\|^2_{\wedge^d Z_0} = \|\Phi_{0,1} \wedge \dots \wedge \Phi_{0,d}\|^2_{\wedge^d Z_0} \exp \int_0^t \operatorname{Tr}(\mathbf{B}_{\tilde{U}(s),\Sigma_d(s),\delta})\, ds. \tag{3.10}$$

For $j \in \mathbb{N}$, define the quantities

$$p_j := \sup\left\{ \operatorname{Tr}(\mathbf{B}_{\tilde{U},\Sigma_j,\delta}) \mid \tilde{U} \in \mathcal{I}_\delta,\, \Sigma_j \subset Z_0,\, \dim \Sigma_j = j \right\}. \tag{3.11}$$

It follows from the results in [21, Ch. V, pp 287–291] that if for some d one has $p_d < 0$ then the Hausdorff dimension of $\mathcal{I}_\delta$ in Z_0 is finite and less than or equal to d, and the fractal dimension of $\mathcal{I}_\delta$ in Z_0 is finite and less than or equal to $d \max_{1 \le j \le d-1}(1 + (p_j)_+/|p_d|)$. Therefore we must choose $\delta > 0$ in such a way that we can find d such that $p_d < 0$.

First we observe that, given an orthonormal basis $\check{\Phi}_1, \dots, \check{\Phi}_d$ of Σ_d, then

$$\begin{aligned} \operatorname{Tr}(\mathbf{B}_{\tilde{U},\Sigma_d,\delta}) &= \sum_{i=1}^{d} \langle \mathbf{B}_{\tilde{U},\Sigma_d,\delta}\check{\Phi}_i, \check{\Phi}_i\rangle_{Z_0} \\ &= \sum_{i=1}^{d}\left(-2\delta\|\check{\phi}_i\|^2_{H^1_0} - 2(\alpha-\delta)\|\check{\psi}_i\|^2_{L^2} + 2\delta(\alpha-\delta)\langle\check{\phi}_i, \check{\psi}_i\rangle_{L^2} + 2\langle\widehat{\partial_u f}(\tilde{u})\check{\phi}_i, \check{\psi}_i\rangle_{L^2}\right), \end{aligned} \tag{3.12}$$

where $\check{\Phi}_i = (\check{\phi}_i, \check{\psi}_i)$, $i = 1, \dots, d$. Now, following the arguments of [20], we choose $\delta := \lambda_1\alpha/(\alpha^2 + 4\lambda_1)$. With this choice of δ, using Cauchy-Schwartz and Young's inequalities and setting

$$\nu_\alpha := \frac{\lambda_1\alpha}{\sqrt{\alpha^2 + 4\lambda_1}(\alpha + \sqrt{\alpha^2 + 4\lambda_1})}, \tag{3.13}$$

we get

$$\operatorname{Tr}(\mathbf{B}_{\tilde{U},\Sigma_d,\delta}) \le -2\nu_\alpha d + \sum_{i=1}^{d}\left(-\alpha\|\check{\psi}_i\|^2_{L^2} + 2\langle\widehat{\partial_u f}(\tilde{u})\check{\phi}_i, \check{\psi}_i\rangle_{L^2}\right); \tag{3.14}$$

using again Cauchy-Schwartz and Young's inequalities, we finally obtain

$$\mathrm{Tr}(\mathbf{B}_{\tilde{U},\Sigma_d,\delta}) \leq -2\nu_\alpha d + \frac{1}{\alpha}\sum_{i=1}^{d} \|\widehat{\partial_u f}(\tilde{u})\check{\phi}_i\|_{L^2}^2. \tag{3.15}$$

Remark 3.2. *Our choice of δ, according to* [20], *is better than the classical $0 < \delta \leq \min\{\alpha/4, \lambda_1/2\alpha\}$ (see e.g.* [21]*): indeed, when considering attractors of dissipative wave equations, it yields dimensional bounds which are independent of α.*

In order to prove finite dimensionality of $\mathcal{I}_\delta$, we have now to find d sufficiently large, so that the right hand side of (3.15) is negative, uniformly with respect to $\tilde{U}$ and Σ_δ. We introduce the following fundamental Hypothesis:

Hypothesis 3.3.

(1) *$\partial_u f(x, 0) \geq 0$ for a.e. $x \in \Omega$;*
(2) *there exists $r > 3$ such that $\partial_u f(\cdot, 0) \in L^r(\Omega)$.*

Notice that property (1) is not really a restriction, since the negative part of $\partial_u f(\cdot, 0)$ can be absorbed by $\beta(\cdot)$.

We observe that, by Hypotheses 2.6 and 3.3, we have:

$$|\partial_u f(x, u)| \leq \partial_u f(x, 0) + C(1 + |u|)|u|, \quad (x, u) \in \Omega \times \mathbb{R}. \tag{3.16}$$

Take $\rho \in \mathcal{S}$ (the Schwartz class) with $\rho(x) > 0$ for all $x \in \mathbb{R}^3$ and, for $\epsilon \geq 0$, define

$$W_{\tilde{U}}(x) := \partial_u f(x, 0) + C(1 + |\tilde{u}|_{L^\infty})|\tilde{u}(x)|, \quad x \in \Omega, \tag{3.17}$$

and

$$W_{\tilde{U},\epsilon}(x) := W_{\tilde{U}}(x) + \epsilon\rho(x), \quad x \in \Omega. \tag{3.18}$$

The reason for introducing the correction $\epsilon\rho(x)$ will be made clear later. Notice that $W_{\tilde{U},\epsilon}(\cdot) \in L^r(\Omega)$ for $\epsilon \geq 0$ and $W_{\tilde{U},\epsilon} > 0$ for $x \in \Omega$ and $\epsilon > 0$. Moreover,

$$\|\widehat{\partial_u f}(\tilde{u})u\|_{L^2}^2 \leq \|W_{\tilde{U}} u\|_{L^2}^2 \leq \|W_{\tilde{U},\epsilon} u\|_{L^2}^2, \quad u \in H_0^1(\Omega). \tag{3.19}$$

It follows from Lemma 4.5 in [14] that the assignment $u \mapsto W_{\tilde{U},\epsilon} u$ defines a compact linear operator from $H_0^1(\Omega)$ to $L^2(\Omega)$. Let us define the following operator $S_{\tilde{U},\epsilon} \colon Z_0 \to Z_0$:

$$S_{\tilde{U},\epsilon}(u, v) := (0, W_{\tilde{U},\epsilon} u), \quad U = (u, v) \in Z_0. \tag{3.20}$$

Then $S_{\tilde{U},\epsilon}$ is compact, and the same is true for its adjoint $S^*_{\tilde{U},\epsilon}$. We have

$$\|W_{\tilde{U},\epsilon} u\|_{L^2}^2 = \langle S_{\tilde{U},\epsilon} U, S_{\tilde{U},\epsilon} U\rangle_{Z_0} = \langle S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon} U, U\rangle_{Z_0}, \quad U = (u, v) \in Z_0. \tag{3.21}$$

The operator $S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon}$ is compact, self-adjoint and non-negative. It follows that its spectrum is

$$\sigma(S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon}) = \{0\} \cup \{\mu_{\tilde{U},\epsilon,j} \mid j = 1, 2, 3, \dots\}, \tag{3.22}$$

where $(\mu_{\tilde{U},\epsilon,j})_{j\in\mathbb{N}}$ is a non-increasing sequence of real numbers tending to 0. The numbers $\mu_{\tilde{U},\epsilon,j}$, $j \in \mathbb{N}$, are the eigenvalues of $S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon}$, repeated according to their multiplicity. In principle, the sequence $(\mu_{\tilde{U},\epsilon,j})_{j\in\mathbb{N}}$ can be ultimately null, but we

shall see that this is not the case. Finally, the sequence $(\mu_{\tilde{U},\epsilon,j})_{j\in\mathbb{N}}$ is characterized by the min – max formulae:

$$\mu_{\tilde{U},\epsilon,j+1} = \min_{\dim E\le j} \max_{\substack{U\in\mathcal{E}^\perp\\ \|U\|_{Z_0}=1}} \langle S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon} U, U\rangle_{Z_0}. \tag{3.23}$$

Let P_Σ be the Z_0-orthogonal projection onto Σ. Arguing as in the proof of Theorem XIII.3 in [18], we obtain

$$\sum_{i=1}^{d} \|\widehat{\partial_u f}(\tilde{u})\check{\phi}_i\|^2_{L^2} \le \sum_{i=1}^{d} \langle S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon} \check{\Phi}_i, \check{\Phi}_i\rangle_{Z_0} = \mathrm{Tr}(P_\Sigma \circ (S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon})|_\Sigma) \le \sum_{i=1}^{d} \mu_{\tilde{U},\epsilon,i}. \tag{3.24}$$

It follows from (3.15) and (3.24) that

$$\mathrm{Tr}(\mathbf{B}_{\tilde{U},\Sigma_d,\delta}) \le -\frac{d}{\alpha}\left(2\nu_\alpha\alpha - \frac{1}{d}\sum_{i=1}^{d}\mu_{\tilde{U},\epsilon,i}\right). \tag{3.25}$$

Now, since $\mu_{\tilde{U},\epsilon,i} \to 0$ as $i \to \infty$, the also the *Cesaro means* $(1/d)\sum_{i=1}^{d}\mu_{\tilde{U},\epsilon,i} \to 0$ as $d \to \infty$. Therefore there exists $d = d(\tilde{U})$ such that the right-hand side of (3.25) is negative. The problem is that $d(\tilde{U})$ depends on $\tilde{U}$, so we must perform a more careful inspection of the asymptotic behavior of the sequence $(\mu_{\tilde{U},\epsilon,j})_{j\in\mathbb{N}}$.

Let (μ, Φ) be an eigenvalue-eigenvector pair of $S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon}$, with $\mu \ne 0$. This is equivalent to say that

$$\langle S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon}\Phi, U\rangle_{Z_0} = \mu\langle \Phi, U\rangle_{Z_0} \quad \text{for all } U \in Z_0. \tag{3.26}$$

More explicitly, (3.26) reads

$$\int_\Omega W_{\tilde{U},\epsilon}(x)^2\phi u\, dx = \mu\left(\int_\Omega \nabla\phi\cdot\nabla u\, dx + \int_\Omega \beta(x)\phi u\, dx + \int_\Omega \psi v\, dx\right) \tag{3.27}$$

for all $U \in Z_0$, where $\Phi = (\phi, \psi)$ and $U = (u, v)$. Choosing first $u = 0$ and letting $v \in L^2(\Omega)$ be arbitrary, we get that $\psi = 0$. It follows that ϕ must satisfy

$$\int_\Omega W_{\tilde{U},\epsilon}(x)^2\phi u\, dx = \mu\left(\int_\Omega \nabla\phi\cdot\nabla u\, dx + \int_\Omega \beta(x)\phi u\, dx\right) \quad \text{for all } u \in H^1_0. \tag{3.28}$$

Thus we have obtained that (μ, Φ) is an eigenvalue-eigenvector pair of $S^*_{\tilde{U},\epsilon} S_{\tilde{U},\epsilon}$ with $\mu \ne 0$ if and only if $\psi = 0$ and $(\mu, \phi) = (\lambda^{-1}, \phi)$, where (λ, ϕ) is an eigenvalue-eigenvector pair of the *weighted eigenvalue problem*

$$\int_\Omega \nabla\phi\cdot\nabla u\, dx + \int_\Omega \beta(x)\phi u\, dx = \lambda\int_\Omega W_{\tilde{U},\epsilon}(x)^2\phi u\, dx \quad \text{for all } u \in H^1_0(\Omega). \tag{3.29}$$

In order to study (3.29) we proceed as in [10]: we denote by $L^2_{W_{\tilde{U}\epsilon}}(\Omega)$ the closure of $H^1_0(\Omega)$ with respect to the scalar product

$$\langle u_1, u_2\rangle_{L^2_{W_{\tilde{U},\epsilon}}} := \int_\Omega W_{\tilde{U},\epsilon}(x)^2 u_1 u_2\, dx. \tag{3.30}$$

It turns out that $L^2_{W_{\tilde U,\epsilon}}(\Omega)$ is a separable Hilbert space, and $H^1_0(\Omega)$ is compactly embedded into $L^2_{W_{\tilde U,\epsilon}}(\Omega)$. This is a consequence of the fact that $W^2_{\tilde U,\epsilon} \in L^{r/2}(\Omega)$ with $r > 3$ and $W_{\tilde U,\epsilon}(x) > 0$ a.e. in Ω. The latter observation makes clear the reason for which we introduced the correction $\epsilon\rho(x)$. It follows from the general theory of self-adjoint operators with compact resolvent (see e.g. [5]) that the eigenvalues of (3.29), counted according to their multiplicity, form a sequence $(\lambda_{\tilde U,\epsilon,j})_{j\in\mathbb{N}}$, with $\lambda_{\tilde U,\epsilon,j} \to +\infty$ as $j \to \infty$. Now let $\tilde\lambda > 0$; we want to find an estimate for the number $\mathcal{N}(W_{\tilde U,\epsilon}, \tilde\lambda)$ of eigenvalues of (3.29) which are strictly smaller than $\tilde\lambda$. To this end, we exploit a trick due to Li and Yau (see [12, Cor. 2]). Namely, we notice that, for $\phi \in H^1_0(\Omega)$,

$$\begin{aligned}(3.31)\quad & \frac{\int_\Omega |\nabla\phi|^2\,dx + \int_\Omega \beta(x)\phi^2\,dx - \int_\Omega \tilde\lambda W_{\tilde U,\epsilon}(x)^2\phi^2\,dx}{\int_\Omega \phi^2\,dx} \\ &= \frac{\int_\Omega W_{\tilde U,\epsilon}(x)^2\phi^2\,dx}{\int_\Omega \phi^2\,dx}\left(\frac{\int_\Omega |\nabla\phi|^2\,dx + \int_\Omega \beta(x)\phi^2\,dx}{\int_\Omega W_{\tilde U,\epsilon}(x)^2\phi^2\,dx} - \tilde\lambda\right).\end{aligned}$$

It follows that, given a finite dimensional subspace E of $H^1_0(\Omega)$, the expression on the left-hand side in (3.31) is negative on E if and only if the expression on the right-hand side (3.31) is negative on E. Now we observe that the mapping $u \mapsto -\tilde\lambda W_{\tilde U,\epsilon}(x)^2 u$ is a relatively compact perturbation of $-\Delta + \beta(x)$. Therefore, by Weyl's Theorem, the essential spectrum of $-\Delta + \beta(x) - \tilde\lambda W_{\tilde U,\epsilon}(x)^2$ is contained in $[\lambda_1, +\infty[$. Then, thanks to the min − max characterization of the eigenvalues of self-adjoint operators (see e.g. [18]), we deduce that

$$(3.32)\qquad \mathcal{N}(W_{\tilde U,\epsilon}, \tilde\lambda) = n(-\Delta + \beta(x) - \tilde\lambda W_{\tilde U,\epsilon}(x)^2),$$

where $n(-\Delta+\beta(x)-\tilde\lambda W_{\tilde U,\epsilon}(x)^2)$ is the number of negative eigenvalues of the operator $-\Delta + \beta(x) - \tilde\lambda W_{\tilde U,\epsilon}(x)^2$. The latter can be estimated by mean of Cwickel-Lieb-Rozenblum inequality in its abstract formulation due to Rozenblum and Solomyak (see [19]). Namely, we have

$$(3.33)\qquad n(-\Delta + \beta(x) - \tilde\lambda W_{\tilde U,\epsilon}(x)^2) \le M_r \int_\Omega (\tilde\lambda W_{\tilde U,\epsilon}(x)^2)^{r/2}\,dx,$$

where M_r is an constant depending only on r, λ_1, $|\beta|_{L^q_u}$, and on the constant of the embedding $H^2(\Omega) \subset L^\infty(\Omega)$ (see also [14, Sect. 5] for details; we stress that the constant M_r can be computed explicitly, though the determination of its optimal value seems out of reach). We have thus obtained that

$$(3.34)\qquad \mathcal{N}(W_{\tilde U,\epsilon}, \tilde\lambda) \le \tilde\lambda^{r/2} M_r \int_\Omega W_{\tilde U,\epsilon}(x)^r\,dx.$$

Now fix $j \in \mathbb{N}$. For $\tilde\lambda > \lambda_{\tilde U,\epsilon,j}$ we have

$$(3.35)\qquad j \le N(W_{\tilde U,\epsilon}, \tilde\lambda) \le \tilde\lambda^{r/2} M_r \int_\Omega W_{\tilde U,\epsilon}(x)^r\,dx.$$

By letting $\tilde{\lambda}$ tend to $\lambda_{\tilde{U},\epsilon,j}$ we get

$$j \le \lambda_{\tilde{U},\epsilon,j}^{r/2} M_r \int_\Omega W_{\tilde{U},\epsilon}(x)^r \, dx. \tag{3.36}$$

It follows that

$$\lambda_{\tilde{U},\epsilon,j} \ge M_r^{-2/r} \|W_{\tilde{U},\epsilon}\|_{L^r}^{-2} \, j^{2/r}, \tag{3.37}$$

whence

$$\mu_{\tilde{U},\epsilon,j} \le M_r^{2/r} \|W_{\tilde{U},\epsilon}\|_{L^r}^{2} \, j^{-2/r}. \tag{3.38}$$

Putting together (3.25) and (3.38), we get

$$\operatorname{Tr}(\mathbf{B}_{\tilde{U},\Sigma_d,\delta}) \le -\frac{d}{\alpha}\left(2\nu_\alpha\alpha - \frac{1}{d}\sum_{j=1}^{d} M_r^{2/r}\|W_{\tilde{U},\epsilon}\|_{L^r}^2 \, j^{-2/r}\right). \tag{3.39}$$

Letting ϵ tend to 0 and taking into account (3.17), we finally get

$$\operatorname{Tr}(\mathbf{B}_{\tilde{U},\Sigma_d,\delta}) \le -\frac{M_r^{2/r}\tilde{C}(\mathcal{I})^2 d}{\alpha}\left(\frac{2\nu_\alpha\alpha}{M_r^{2/r}\tilde{C}(\mathcal{I})^2} - \frac{1}{d}\sum_{j=1}^{d} j^{-2/r}\right), \tag{3.40}$$

where

$$\tilde{C}(\mathcal{I}) := \|\partial_u f(\cdot,0)\|_{L^r} + C(1 + \sup_{(u,v)\in\mathcal{I}} \|u\|_{L^\infty}) \sup_{(u,v)\in\mathcal{I}} \|u\|_{L^r}. \tag{3.41}$$

We have thus obtained an estimate for $\operatorname{Tr}(\mathbf{B}_{\tilde{U},\Sigma_d,\delta})$ which is uniform with respect to $\tilde{U}$ and Σ_d. Now we are in a position to state and prove the main result of the paper:

Theorem 3.4. *Assume Hypotheses 2.2, 2.5, 2.6 and 3.3 are satisfied. Let $\mathcal{I} \subset Z_0$ be a compact invariant set of the semiflow $\Pi(t)$ generated by (2.14). Let ν_α and $\tilde{C}(\mathcal{I})$ be defined by (3.13) and (3.41) respectively, and let M_r be the constant of the Cwickel-Lieb-Rozenblum inequality (3.33). Let $d > 0$ be such that*

$$\frac{1}{d}\sum_{j=1}^{d} j^{-2/r} \le \frac{\nu_\alpha\alpha}{M_r^{2/r}\tilde{C}(\mathcal{I})^2}. \tag{3.42}$$

Then the Hausdorff (resp. the fractal) dimension of $\mathcal{I}$ in Z_0 is finite, and is less than or equal to d (resp. $2d$).

Proof. Let p_j, $j \in \mathbb{N}$, be the numbers defined by (3.11). If d satisfies condition (3.42), then (3.40) implies that $p_d \le -\nu_\alpha d$. Moreover, for $j = 1, \ldots, d-1$, one has

$$(p_j)_+ \le \frac{M_r^{2/r}\tilde{C}(\mathcal{I})^2}{\alpha}\sum_{i=1}^{j-1} i^{-2r} \le \frac{M_r^{2/r}\tilde{C}(\mathcal{I})^2}{\alpha}\sum_{i=1}^{d} i^{-2r} \le \nu_\alpha d.$$

It follows from Proposition 2.12 and from the results in [21, Ch. V, pp 287–291] that $\dim_H(\mathcal{I}) \le d$ and $\dim_F(\mathcal{I}) \le d\max_{1\le j\le d-1}(1 + (p_j)_+/|p_d|) \le 2d$. $\square$

Remark 3.5. *We can give an explicit estimate of d just noticing that*

$$\frac{1}{d}\sum_{i=1}^{d} i^{-2r} \leq \frac{1}{d}\int_0^d s^{-2/r}\,ds = \frac{r}{r-2}d^{-2/r}. \tag{3.43}$$

It follows that

$$\dim_H(\mathcal{I}) \leq \left(\frac{r}{r-2}\frac{M_r^{2/r}\tilde{C}(\mathcal{I})^2}{\nu_\alpha \alpha}\right)^{r/2} \tag{3.44}$$

and

$$\dim_F(\mathcal{I}) \leq 2\left(\frac{r}{r-2}\frac{M_r^{2/r}\tilde{C}(\mathcal{I})^2}{\nu_\alpha \alpha}\right)^{r/2}. \tag{3.45}$$

Notice that $\nu_\alpha\alpha \to \lambda_1$ as $\alpha \to \infty$. Therefore, if we have a family $\mathcal{I}_\alpha$ of invariant sets of $\Pi(t) = \Pi_\alpha(t)$ and if we can control $|\mathcal{I}_\alpha|_{Z^1}$ independently of α, we obtain that the dimension of $\mathcal{I}_\alpha$ remains bounded as $\alpha \to \infty$. This is actually the case when the non-linearity f is dissipative and $\mathcal{I}_\alpha$ is the compact global attractor of $\Pi_\alpha(t)$, as we shall see in the next section.

4. Dissipative equations: dimension of the attractor

In this section we consider the equation

$$\begin{aligned} \epsilon u_{tt} + \ u_t + \beta(x)u - \Delta u = f(x,u), &\quad (t,x) \in [0,+\infty[\times\Omega, \\ u = 0, &\quad (t,x) \in [0,+\infty[\times\partial\Omega \end{aligned} \tag{4.1}$$

where $\epsilon \in]0,1]$. Besides Hypotheses 2.2, 2.5, 2.6 and 3.3, we assume:

Hypothesis 4.1. *There exists a positive number μ and a function $c(\cdot) \in L^1(\Omega)$ such that:*

1. $f(x,u)u - \mu F(x,u) \leq c(x)$;
2. $F(x,u) \leq c(x)$.

Here, $F(x,u) := \int_0^u f(x,s)\,ds$, $(x,u) \in \Omega \times \mathbb{R}$.

It was proved in [16] that, under Hypotheses 2.2, 2.6 and 4.1, for every $\epsilon \in]0,1]$ equation (4.1) generates a global semiflow in Z_0, possessing a compact global attractor $\mathcal{A}_\epsilon$. Moreover, there exists a positive constant R such that

$$\sup_{\epsilon\in]0,1]} \sup\{\|u\|^2_{H_0^1} + \epsilon\|v\|^2_{L^2} \mid (u,v) \in \mathcal{A}_\epsilon\} \leq R.$$

The constant R depends only on the constants in Hypotheses 2.2, 2.6 and 4.1 and on $\|c(\cdot)\|_{L^1}$, and can be explicitly computed (see [16]). In particular, R is independent of ϵ. Moreover, it was proved in [13] that there exists a positive constant $\tilde{R}$ such that

$$\sup_{\epsilon\in]0,1]} \sup\{\|u\|^2_{H^2\cap H_0^1} + \|v\|^2_{H_0^1} \mid (u,v) \in \mathcal{A}_\epsilon\} \leq \tilde{R}.$$

Also, the constant $\tilde{R}$ depends only on the constants in Hypotheses 2.2, 2.6 and 4.1 and on $\|c(\cdot)\|_{L^1}$ and can be explicitly computed (see [13]). In particular, $\tilde{R}$ is independent of ϵ. By a time re-scaling ($t = \epsilon^{1/2}s$) we see that (4.1) is equivalent to

$$\begin{aligned} \epsilon\check{u}_{ss} + \alpha\ \check{u}_s + \beta(x)\check{u} - \Delta\check{u} &= f(x,\check{u}), && (s,x) \in [0,+\infty[\times\Omega, \\ \check{u} &= 0, && (s,x) \in [0,+\infty[\times\partial\Omega \end{aligned} \tag{4.2}$$

where $\alpha := \epsilon^{-1/2}$. Equation (4.2) possesses a compact global attractor $\check{\mathcal{A}}_\alpha$, such that

$$\check{\mathcal{A}}_\alpha = \{(\check{u},\check{v}) \in Z_0 \mid (\check{u},\alpha\check{v}) \in \mathcal{A}_{\alpha^{-2}}\}. \tag{4.3}$$

It follows that $|\check{\mathcal{A}}_\alpha|_{Z_0} \le R$ and $|\check{\mathcal{A}}_\alpha|_{Z_1} \le \tilde{R}$. As a consequence, the constant $\tilde{C}(\check{\mathcal{A}}_\alpha)$ in (3.44) and (3.45) can be explicitly computed, and in particular it is independent of α. We have then

$$\dim_H(\mathcal{A}_\epsilon) = \dim_H(\check{\mathcal{A}}_{\epsilon^{-1/2}}) \le \left(\frac{r}{r-2}\frac{M_r^{2/r}\tilde{C}(\check{\mathcal{A}}_{\epsilon^{-1/2}})^2}{\nu_{\epsilon^{-1/2}}\epsilon^{-1/2}}\right)^{r/2} \tag{4.4}$$

and

$$\dim_F(\mathcal{A}_\epsilon) = \dim_F(\check{\mathcal{A}}_{\epsilon^{-1/2}}) \le 2\left(\frac{r}{r-2}\frac{M_r^{2/r}\tilde{C}(\check{\mathcal{A}}_{\epsilon^{-1/2}})^2}{\nu_{\epsilon^{-1/2}}\epsilon^{-1/2}}\right)^{r/2}. \tag{4.5}$$

Since $\nu_\alpha\alpha \to \lambda_1$ as $\alpha \to \infty$, we obtain that $\dim_H(\mathcal{A}_\epsilon)$ and $\dim_F(\mathcal{A}_\epsilon)$ remain bounded as $\epsilon \to 0$, coherently with the fact that the $\mathcal{A}_\epsilon$ "converge", as $\epsilon \to 0$, to the attractor of the parabolic equation

$$\begin{aligned} u_t + \beta(x)u - \Delta u &= f(x,u), && (t,x) \in [0,+\infty[\times\Omega, \\ u &= 0, && (t,x) \in [0,+\infty[\times\partial\Omega \end{aligned} \tag{4.6}$$

(see [17] and [13]).

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